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Weak P-subsets of Stone spaces

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Abstract

Let $\mathcal{A}$, $\mathcal{B}$ be complete Boolean algebras of size $\mathfrak{c}$, and let $\mathrm{st}(\mathcal{A}), \mathrm{st}(\mathcal{B})$ be their Stone spaces. We describe conditions which imply that $\mathrm{st}(\mathcal{A})$ is homeomorphic to a weak P-subset of $\mathrm{st}(\mathcal{B})$.

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1. Introduction

In this paper, $\mathrm{st}(\mathcal{B})$ denotes the Stone space of the Boolean algebra $\mathcal{B}$, and $N_b \subset \mathrm{st}(\mathcal{B})$ (for $b \in \mathcal{B}$) is the basic clopen set $\{p \in \mathrm{st}(\mathcal{B})\colon\ b \in p\}$. $\mathbb{N}^*$ is the space $\mathrm{st}(\mathcal{P}(\omega)/\mathit{fin})$.

Given a filter $\mathcal{F}$ over the Boolean algebra $\mathcal{B}$, let $\mathcal{F}^*$ denote its dual ideal and let $\mathcal{K}_{\mathcal{F}} = \bigcap_{b\in\mathcal{F}} N_b \subset \mathrm{st}(\mathcal{B})$. So, $\mathcal{K}_{\mathcal{F}}$ is the set of all ultrafilters over $\mathcal{B}$ which extend $\mathcal{F}$.

The symbol $\tilde{}$ will be used to denote the complement, either of an object in a Boolean algebra or of a set in a topological space.

Before stating the main theorem of this paper, we need several definitions.

Definition 1.1. Let $\mathcal{X}$ be any topological space. A closed subset $\mathcal{Y}$ of $\mathcal{X}$ is called a *weak* P-*subset* of $\mathcal{X}$ iff: whenever $\{x_n\colon\ n \in \omega\} \subset \mathcal{X}\setminus\mathcal{Y}$ then $\mathrm{cl}(\{x_n\colon\ n \in \omega\}) \cap \mathcal{Y} = \emptyset$. If $\mathcal{Y} = \{y\}$ then y is called a weak P-point.

When finding weak P-subsets, it is often useful to refer to the stronger property of being κ-OK for some cardinal $\kappa \geqslant \omega_1$. Kunen [7] formulated this concept and used it to construct weak P-points in $\mathbb{N}^*$.

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Definition 1.2. A closed subset $\mathcal{Y}$ of $\mathcal{X}$ is *κ-OK* iff: whenever $U_0, U_1, U_2, \ldots$ are open supersets of $\mathcal{Y}$, then there are open supersets V_ζ $(\zeta < \kappa)$ of $\mathcal{Y}$ such that $\forall m < \omega\ \forall \zeta_1 < \zeta_2 < \cdots < \zeta_m < \kappa\ (V_{\zeta_1} \cap V_{\zeta_2} \cap \cdots \cap V_{\zeta_m} \subset U_m)$.

Lemma 1.3. *If $\mathcal{X}$ is T_1 and the closed subset $\mathcal{Y}$ is ω_1-OK in $\mathcal{X}$, then $\mathcal{Y}$ is a weak* P*-subset of $\mathcal{X}$.*

The proof of this is exactly as in [7, p. 743]. Note that κ-OK only gets stronger as κ gets larger, so a closed subset which is κ-OK for some $\kappa \geqslant \omega_1$ will also be ω_1-OK. From now on, we will be taking $\kappa = \mathfrak{c}$.

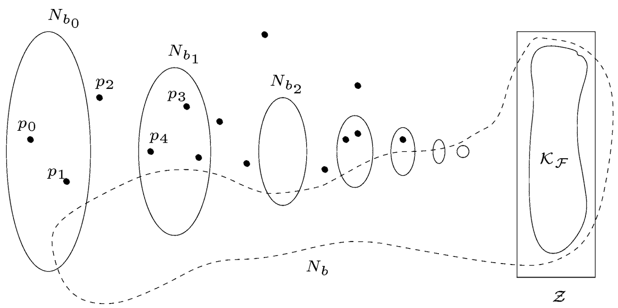
For most of the remainder of this paper, we will be restricting our attention to extremally disconnected spaces, i.e., Stone spaces of complete Boolean algebras. (In the last section, we will also consider Boolean algebras which have the countable separation property.) Compare the following definition with the one given by van Mill [10] for normal topological spaces. For the sake of simplicity, we have specialized this definition to Boolean algebras.

Definition 1.4. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of the complete Boolean algebra $\mathcal{B}$. Then a filter $\mathcal{F}$ over $\mathcal{B}$ is *nice* over $\{b_n\colon n \in \omega\}$ iff:

(1) $\bigvee_{n<\omega} b_n \in \mathcal{F}$.
(2) $\forall n < \omega\ (\widetilde{b_n} \in \mathcal{F})$.
(3) $\forall b \in \mathcal{F}\ (\{n\colon b \wedge b_n = 0_{\mathcal{B}}\}$ is finite$)$.

In this definition, (1) says that $\mathcal{K}_{\mathcal{F}} \subset N_{\bigvee_n b_n}$ and (2) says that $\forall n < \omega\ (\mathcal{K}_{\mathcal{F}} \cap N_{b_n} = \emptyset)$. So together we have $\mathcal{K}_{\mathcal{F}} \subset N_{\bigvee_n b_n} \setminus \bigcup_n N_{b_n}$. (3) says that in some sense, $\mathcal{K}_{\mathcal{F}}$ is big enough; or alternately, $\mathcal{F}$ is not very strict.

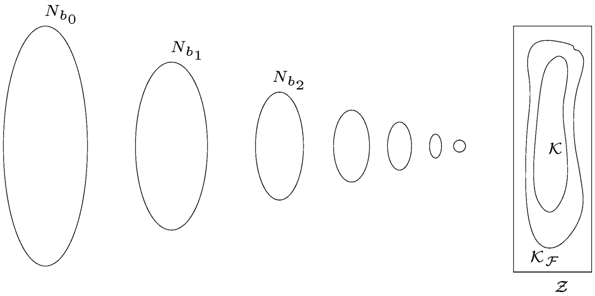
Definition 1.5. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of the complete Boolean algebra $\mathcal{B}$. Set $\mathcal{Z} := N_{\bigvee_n b_n} \setminus \bigcup_n N_{b_n} \subset \mathrm{st}(\mathcal{B})$. A nice filter $\mathcal{F}$ over $\{b_n\colon n \in w\}$ is said to *miss countable sets* iff: whenever p_n $(n \in \omega)$ are ultrafilters over $\mathcal{B}$ and $\mathcal{Z} \cap \{p_n\colon n \in \omega\} = \emptyset$, then $\mathcal{K}_{\mathcal{F}} \cap \mathrm{cl}(\{p_n\colon n \in \omega\}) = \emptyset$ (i.e., $\exists b \in \mathcal{F}\ \forall n\ (b \notin p_n)$).



Later, *nice filter* and *misses countable sets* will be defined for general Boolean algebras. When a Boolean algebra is complete, the new definitions will be equivalent to these ones.

Now we can state the main theorem, along with its corollary.

Theorem 1.6 (Main Theorem). *Assume that $\mathcal{A}$ and $\mathcal{B}$ are complete Boolean algebras of cardinality $\mathfrak{c}$. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of $\mathcal{B}$, and set $\mathcal{Z} := N_{\bigvee_n b_n} \setminus \bigcup_n N_{b_n}$. Let $\mathcal{F}$ be a nice filter over $\{b_n\colon n \in \omega\}$. Then $\mathrm{st}(\mathcal{A})$ is homeomorphic to some $\mathcal{K} \subset \mathcal{K}_{\mathcal{F}}$ such that $\mathcal{K}$ is $\mathfrak{c}$-OK in $\mathcal{Z}$.*



Corollary 1.7. *If $\mathcal{A}$, $\mathcal{B}$, and $\{b_n\colon n \in \omega\}$ are as in Theorem* 1.6, *and if also $\mathcal{B}$ contains a nice filter $\mathcal{F}$ over $\{b_n\colon n \in \omega\}$ which misses countable sets, then* $\mathrm{st}(\mathcal{A})$ *is homeomorphic to a weak* P*-subset of* $\mathrm{st}(\mathcal{B})$.

Proof. Apply Theorem 1.6 using the nice filter $\mathcal{F}$, and let $\mathcal{K}$ be as in the theorem, so $\mathrm{st}(\mathcal{A})$ is homeomorphic to $\mathcal{K} \subset \mathcal{K}_{\mathcal{F}}$. Let $X = \{x_n\colon n \in \omega\} \subset \mathrm{st}(\mathcal{B}) \setminus \mathcal{K}$. To show that $\mathrm{cl}(X) \cap \mathcal{K} = \emptyset$, we consider $X \cap \widetilde{\mathcal{Z}}$ and $X \cap \mathcal{Z}$ separately. Since $\mathcal{K} \subset \mathcal{K}_{\mathcal{F}}$ and since $\mathcal{F}$ misses countable sets, $\mathrm{cl}(X \cap \widetilde{\mathcal{Z}}) \cap \mathcal{K} \subset \mathrm{cl}(X \cap \widetilde{\mathcal{Z}}) \cap \mathcal{K}_{\mathcal{F}} = \emptyset$. Also since $\mathcal{K}$ is a $\mathfrak{c}$-OK subset of $\mathcal{Z}$, we know that $\mathcal{K}$ is a weak P-subset of $\mathcal{Z}$, so $\mathrm{cl}(X \cap \mathcal{Z}) \cap \mathcal{K} = \emptyset$. Therefore $\mathrm{cl}(X) \cap \mathcal{K} = \emptyset$, as desired. □

2. Some observations

It is worth remarking on some of the ways that Theorem 1.6 and its corollary are similar to and different from several previous results. We would also like to mention an application of weak P-subsets. Then we discuss what is known concerning which Boolean algebras Corollary 1.7 can be applied to.

2.1. Previous results

First, note that in Theorem 1.6, if $\mathcal{F}$ is just the cofinite filter over $\{b_n\colon n \in \omega\}$, then $\mathcal{K}_{\mathcal{F}} = \mathcal{Z}$ and we get that $\mathrm{st}(\mathcal{A})$ is homeomorphic to a $\mathfrak{c}$-OK subset of $\mathcal{Z}$. In the specific case that $\mathcal{B} = \mathcal{P}(\omega)$, $b_n = \{n\}$, and $\mathcal{F}$ is the cofinite filter over ω, then $\mathcal{Z} = \mathbb{N}^*$. This instance of Theorem 1.6 is already known, due to Simon [9]:

Theorem 2.1. *If $\mathcal{A}$ is a complete Boolean algebra of cardinality $\mathfrak{c}$, then* $\mathrm{st}(\mathcal{A})$ *can be embedded as a $\mathfrak{c}$-OK subset of* $\mathbb{N}^*$.

To prove this theorem, Simon proves an equivalent statement: if $\mathcal{A}$ is a complete Boolean algebra and $|\mathcal{A}| = \mathfrak{c}$, then there is a homomorphism $h : \mathcal{P}(\omega)/\mathit{fin} \twoheadrightarrow \mathcal{A}$ which satisfies certain special properties. (The purpose of these properties is to achieve the $\mathfrak{c}$-OK part of the theorem.) He constructs the homomorphism h by transfinite induction. Many of the same ideas will be instrumental in proving Theorem 1.6, which may be seen as a generalization of Simon's theorem.

The following theorem and corollary are due to van Mill [11]. Here, $\mathcal{X}^*$ denotes $\beta\mathcal{X} - \mathcal{X}$. We refer the reader to van Mill's paper for the other definitions involved, including his version of *nice*.

Theorem 2.2. *Let $\mathcal{X} = \omega \times \mathcal{Z}$ where $\mathcal{Z}$ is a compact space of weight at most $\mathfrak{c}$, and suppose that $\mathcal{F}$ is a nice filter on $\mathcal{X}$. If $\mathcal{Y}$ is a continuous image of $\mathbb{N}^*$, then there is a continuous surjection $g : \mathcal{X}^* \to \mathcal{Y}$ and a closed $\mathfrak{c}$-OK set $\mathcal{K} \subset \mathcal{X}^*$ such that $\mathcal{K} \subset \bigcap_{F \in \mathcal{F}} F^*$ and $g \restriction \mathcal{K}$ is irreducible.*

Corollary 2.3. *If, in Theorem* 2.2*, $\mathcal{Y}$ has the ccc, then the projective cover $E\mathcal{Y}$ of $\mathcal{Y}$ embeds in $\mathcal{X}^*$ as a $\mathfrak{c}$-OK set.*

Taken together, these results are quite similar to Theorem 1.6. For each n identify $\{n\} \times \mathcal{Z}$ with N_{b_n}, so $\mathcal{X}$ plays the role of $\bigcup_n N_{b_n}$ and $\mathcal{X}^*$ the role of $\mathcal{Z} = N_{\bigvee_n b_n} \setminus \bigcup_n N_{b_n}$. Also, $\mathcal{Y}$ represents $\mathrm{st}(\mathcal{A})$, and therefore $\mathcal{Y} = E\mathcal{Y}$. With these substitutions, however, Theorem 1.6 does not entirely follow from Theorem 2.2. A superficial problem is that the N_{b_n} may not be homeomorphic to each other, which means we cannot actually identify each with $\{n\} \times \mathcal{Z}$ for a fixed $\mathcal{Z}$. But to carry out van Mill's proof, it appears that $\mathcal{X}$ does not really need to be a sum of homeomorphic spaces, so putting $\mathcal{X} = \bigcup_n N_{b_n}$ is probably fine.

The main difference is this: to apply Theorem 2.2, we need to have that $\mathrm{st}(\mathcal{A})$ is a continuous image of $\mathbb{N}^*$, in other words (by Stone duality) that the Boolean algebra $\mathcal{A}$ maps isomorphically into $\mathcal{P}(\omega)/\mathit{fin}$. Since we are taking $\mathcal{A}$ to have cardinality $\mathfrak{c}$, this is true under CH. But it's not true in ZFC that a complete Boolean algebra of size $\mathfrak{c}$ always maps into $\mathcal{P}(\omega)/\mathit{fin}$. In fact, Dow and Hart [4] have recently shown that under OCA, the measure algebra of $[0, 1]$ does not embed in $\mathcal{P}(\omega)/\mathit{fin}$. So we see that while van Mill's results are similar to Theorem 1.6, and the proofs involve some of the same techniques, the particular difficulties encountered are different.

Note that in Theorem 2.2 and Corollary 2.3, if the nice filter $\mathcal{F}$ happens to miss countable sets, then $E\mathcal{Y}$ embeds as a weak P-subset of $\beta\mathcal{X}$. This is parallel to Corollary 1.7.

2.2. An application

Why might we want to embed spaces inside each other as weak P-subsets? Aside from being an interesting endeavor in its own right, it enables us to carry over results about some spaces into other spaces.

Definition 2.4. A point $x \in \mathcal{X}$ is *discretely untouchable* iff x is not a limit of any countable discrete subset of $\mathcal{X} \backslash \{x\}$. $Q \subset \mathcal{X}$ is said to have *property* (D) iff Q is countable, dense in itself, and consists of discretely untouchable points of $\mathcal{X}$.

In [6], Kunen proves that under Martin's Axiom, if $\mathcal{B}$ is a nonatomic measure algebra of size $\mathfrak{c}$, then $\mathrm{st}(\mathcal{B})$ contains a subset with property (D). By embedding $\mathrm{st}(\mathcal{B})$ into $\mathbb{N}^*$ as a P-set (using MA again), he shows that $\mathbb{N}^*$ also contains a subset with property (D).

However, using Theorem 1.6—or alternately Theorems 2.1 and 2.2, and Corollary 2.3—one can show that under just ZFC, $\mathbb{N}^*$ and $\mathrm{st}(\mathcal{B})$ both contain subsets with property (D).

First we need a little more background work. Let $\mathrm{ro}(2^\omega)$ denote the regular open algebra of 2^ω (the Cantor set).

Fact 1. $\mathrm{st}(\mathrm{ro}(2^\omega))$ *contains a discretely untouchable point.*

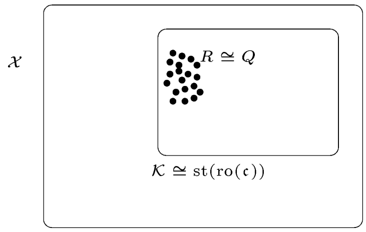
Van Mill shows this in Section 3.3 of [11]. A more general result, due to Balcar and Simon [1], is that every *ccc* extremally disconnected compact space $\mathcal{X}$ such that $\underline{\pi\chi}(\mathcal{X}) = \underline{\pi w}(\mathcal{X}) \leqslant \mathfrak{c}$ contains a discretely untouchable point. Here, $\underline{\pi\chi}(\mathcal{X})$ is the minimal π-character of any point of $\mathcal{X}$, and $\underline{\pi w}(\mathcal{X})$ is the minimal π-weight of any nonempty open subset of $\mathcal{X}$. In the case that $\mathcal{X} = \mathrm{st}(\mathrm{ro}(2^\omega))$, then the collection $\{N_u\colon\ u \text{ clopen} \subset 2^\omega\}$ is countable and forms a π-base for $\mathcal{X}$, so $\underline{\pi\chi}(\mathcal{X}) = \underline{\pi w}(\mathcal{X}) = \omega$.

Fact 2. $\mathrm{st}(\mathrm{ro}(2^\omega))$ *contains a subset with property* (D).

Proof. Whenever u is clopen in 2^ω, then the basic clopen subset N_u of $\mathrm{st}(\mathrm{ro}(2^\omega))$ looks just like $\mathrm{st}(\mathrm{ro}(2^\omega))$ itself, so by Fact 1 we can choose a point q_u which is discretely untouchable in N_u. Let $Q = \{q_u\colon\ u \text{ is clopen in } 2^\omega\}$ be the collection of these points. Then it is easy to check that Q has property (D), i.e., Q is a countable set of discretely untouchable points which is dense in itself. □

Fact 3. *If the space* $\mathcal{X}$ *has a weak* P-*subset which is homeomorphic to* $\mathrm{st}(\mathrm{ro}(2^\omega))$, *then* $\mathcal{X}$ *contains a subset* R *which has property* (D).

Proof. Let $\mathcal{K} \subset \mathcal{X}$ be a weak P-subset which is homeomorphic to $\mathrm{st}(\mathrm{ro}(2^\omega))$. We know that $\mathrm{st}(\mathrm{ro}(2^\omega))$ contains a subset Q which has property (D). Let $R \subset \mathcal{K}$ be the homeomorphic copy of Q. Then R is a countable set of discretely untouchable points of $\mathcal{K}$ which is dense in itself. We need that the points of R are discretely untouchable with respect to the entire space $\mathcal{X}$. But since $\mathcal{K}$ is a weak P-subset of $\mathcal{X}$, no point of R is a limit of *any* countable sequence outside of $\mathcal{K}$. Therefore R has property (D) in $\mathcal{X}$ as well. □



Corollary 2.5 (ZFC).

(1) $\mathbb{N}^*$ *contains a subset with property* (D).

(2) *If* $(\mathcal{B}, \mu)$ *is a nonatomic measure algebra of size* $\mathfrak{c}$, *then* $\mathrm{st}(\mathcal{B})$ *contains a subset which has property* (D).

Proof. The first part of the corollary follows immediately from Theorem 2.1 and Fact 3.

The second part follows from Corollary 1.7 (or alternately Theorem 2.2 and Corollary 2.3) and Fact 3. Whichever we use, we still need to verify that $\mathcal{B}$ contains a nice filter $\mathcal{F}$ which misses countable sets. One way to do this is to cite a result of A. Dow (see the next section for more on this), but here we describe a direct method using the measure μ.

Fix any nonzero disjoint $b_0, b_1, b_2, \ldots \in \mathcal{B}$. Let $\mathcal{F}$ be the filter generated by

$$\left\{b \in \mathcal{B}\colon \forall n\left(\mu(b \wedge b_n) \geqslant \mu(b_n) - \frac{\mu(b_n)}{n+1}\right)\right\}.$$

It is clear that the intersection of any m of these generating elements has nonempty meet with b_n for $n \geqslant m$. Also, since a countable subset of $\mathrm{st}(\mathcal{B})$ can be covered by an element b of $\mathcal{B}$ having arbitrarily small positive measure, $\mathcal{F}$ misses countable sets. Given a countable set, choose $b \in \mathcal{B}$ covering it such that $\mu(b \wedge b_n)$ is small enough (for each n) to ensure that $\tilde{b} \in \mathcal{F}$. Therefore $\mathcal{F}$ gives the desired filter. $\square$

2.3. When does Corollary 1.7 apply?

If, in Theorem 1.6 and Corollary 1.7, we take $\mathcal{A}$ to be the trivial Boolean algebra $\{0, 1\}$, then $\mathrm{st}(\mathcal{A})$ is a singleton and we end up with a weak P-point of $\mathrm{st}(\mathcal{B})$ (assuming that $\mathcal{B}$ has a nice filter which misses countable sets). In this way, Theorem 1.6 may be viewed as a generalization of theorems which produce weak P-points in spaces, except of course that we are just considering certain 0-dimensional spaces.

Since the search for weak P-points has often been reduced to the search for nice filters which miss countable sets, Corollary 1.7 is readily seen to apply to many Boolean algebras which have been shown to contain such filters. In the rest of this section, some of the Boolean algebras under discussion may come in many sizes, but remember that to use Corollary 1.7 the Boolean algebra $\mathcal{B}$ needs to be complete and of size $\mathfrak{c}$. In Section 4 we shall see that in fact the "complete" condition can be weakened to the countable separation property (c.s.p.).

In [5], Dow and van Mill show that when $\mathcal{B}$ is nowhere *ccc* and has the c.s.p., then $\mathcal{B}$ contains a nice filter which misses countable sets. This result basically reduces the problem to the *ccc* case. In [3], Dow shows that a Boolean algebra $\mathcal{B}$ having the c.s.p. contains such a filter whenever the following holds: there are disjoint nonzero elements $b_0, b_1, b_2, \ldots \in \mathcal{B}$ such that the basic clopen sets N_{b_n} $(n \in \omega)$ of $\mathrm{st}(\mathcal{B})$ are *ccc*, nowhere separable, and pairwise homeomorphic. See [3, pp. 558–559] for the construction of the appropriate filter over $\bigcup_n N_{b_n}$ (which corresponds to a filter over $\{b_n\colon n \in \omega\}$). Note that by Maraham's theorem, every nonatomic measure algebra is covered by this case; however, the proof of Corollary 2.5 gives an alternate construction.

Actually, it is difficult to think of *ccc* Boolean algebras having the c.s.p. which are *not* included in this last case. It seems that "most" such Boolean algebras will contain countably many isomorphic elements $b_0, b_1, b_2, \ldots,$ in which case, assuming that $\mathrm{st}(\mathcal{B})$ is nowhere separable, Dow's result applies.

One exception is when $\mathcal{B}$ is Suslin; Suslin algebras may be rigid, implying they do not contain any two isomorphic elements. However, Kunen has pointed out that we can still find a nice filter which misses countable sets. This argument is similar to Dow's proof in [3] that, under the set theoretic assumption $\mathfrak{b} = \mathfrak{c}$, every compact *ccc* nowhere separable F-space contains a weak P-point. (Again, he does this by finding a nice filter which misses countable sets.) When $\mathcal{B}$ is Suslin, we do not need $\mathfrak{b} = \mathfrak{c}$ because we can generate the filter in just ω_1 steps. Fixing disjoint nonzero $b_0, b_1, b_2, \ldots \in \mathcal{B}$, one can construct a special sort of Suslin tree below each b_n. Each level of the tree is a maximal antichain below b_n; these ω_1 antichains are used to inductively construct the generating elements for the filter.

Another interesting Boolean algebra is the Bell algebra described in [2]. This is an example of a *ccc* subalgebra of $\mathcal{P}(\omega)/\mathit{fin}$ which is not σ-centered; in fact, its Stone space is nowhere separable. Since this particular algebra does not have the c.s.p., we consider its completion, say $\mathcal{B}$. It is not clear whether $\mathcal{B}$ contains infinitely many isomorphic elements, but it does contain a nice filter which misses countable sets. Such a filter can be constructed directly, much as the one described for the measure algebra.

3. The proof

Now we turn our attention to the proof of Theorem 1.6. Let $\mathcal{A}, \mathcal{B}, b_0, b_1, b_2, \ldots, \mathcal{Z}$, and $\mathcal{F}$ be as in the hypothesis of the theorem.

Notation. Let $\mathcal{I}$ be the ideal $\{b \in \mathcal{B}\colon \{n\colon b \wedge b_n \neq 0_{\mathcal{B}}\}$ is finite$\}$. For $c, d \in \mathcal{B}$ write $c \leqslant^* d$ to mean that $c \backslash d \in \mathcal{I}$. Then "$c =^* 0_{\mathcal{B}}$" means that $c \in \mathcal{I}$ so c has non-zero meet with only finitely many b_n's, and "$c >^* 0_{\mathcal{B}}$" means that $c \notin \mathcal{I}$ so c has non-zero meet with infinitely many b_n's.

Also, given a homomorphism $h : \mathcal{B} \to \mathcal{A}$, let $h^{-1}(X)$ (for $X \subset \mathcal{A}$) denote the set $\{b \in \mathcal{B}\colon h(b) \in X\}$, and let $h^* : \mathrm{st}(\mathcal{A}) \to \mathrm{st}(\mathcal{B})$ be the map defined by $h^*(p) = h^{-1}(p)\ \forall p \in \mathrm{st}(\mathcal{A})$. Then h^* is a continuous map, and h^* is one-to-one if and only if h is onto (denoted $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$).

To prove Theorem 1.6, we construct a homomorphism $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ which satisfies

Property $(*)$**.** Whenever $\{d_n\colon n \in \omega\} \subset \mathcal{B}$ and $\forall n < \omega (h(d_n) = 1_{\mathcal{A}})$, then there are $y_\eta \in \mathcal{B}$ $(\eta < \mathfrak{c})$ such that:

- each $h(y_\eta) = 1_{\mathcal{A}}$,
- $\forall m\, \forall \eta_1 < \eta_2 < \cdots < \eta_m$ we have $(\mathcal{Z} \cap N_{y_{\eta_1}} \cap \cdots \cap N_{y_{\eta_m}}) \subset (\mathcal{Z} \cap N_{d_m})$, i.e., $y_{\eta_1} \wedge \cdots \wedge y_{\eta_m} \leqslant^* d_m$.

In addition to Property $(*)$, we will specify that $h(b) = 1_{\mathcal{A}}$ for all $b \in \mathcal{F}$. Then $h^*(\mathrm{st}(\mathcal{A})) \subset \mathcal{K}_{\mathcal{F}}$. The map $h^* : \mathrm{st}(\mathcal{A}) \to \mathrm{st}(\mathcal{B})$ gives the desired embedding.

Before beginning, we will need a particular type of independent subset of $\mathcal{B}$. The following definitions are almost straight out of [7].

Definition 3.1. Let $\mathcal{B}$ be a Boolean algebra and let $\mathcal{F}$ be a filter over $\mathcal{B}$. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of $\mathcal{B}$.

(a) Given m such that $0 < m < \omega$, the indexed family $\{a_\zeta\colon \zeta < \kappa\} \subset \mathcal{B}$ is called *precisely m-linked* w.r.t. $\mathcal{F}$ iff:

$$\forall \zeta_1 < \zeta_2 < \cdots < \zeta_m < \zeta_{m+1}\ \forall b \in \mathcal{F}$$
$$\big(\big(b \wedge a_{\zeta_1} \wedge a_{\zeta_2} \wedge \cdots \wedge a_{\zeta_m} >^* 0_{\mathcal{B}}\big) \text{ but } \big(a_{\zeta_1} \wedge \cdots \wedge a_{\zeta_m} \wedge a_{\zeta_{m+1}} =^* 0_{\mathcal{B}}\big)\big).$$

(b) The matrix $\{a_{\zeta,m}\colon \zeta < \kappa,\ 0 < m < \omega\}$ is called a *linked system* w.r.t. $\mathcal{F}$ iff for all m, $\{a_{\zeta,m}\colon \zeta < \kappa\}$ is precisely m-linked w.r.t. $\mathcal{F}$, and for all ζ, $a_{\zeta,1} \leqslant a_{\zeta,2} \leqslant a_{\zeta,3} \leqslant \cdots$, i.e., each column is increasing.

$$\begin{array}{cl} \left(\begin{array}{ccccc} a_{0,1} & a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ a_{0,2} & a_{1,2} & a_{2,2} & a_{3,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ a_{0,m} & a_{1,m} & a_{2,m} & a_{3,m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}\right) & \begin{array}{l} \leftarrow \text{ precisely 1-linked} \\ \leftarrow \text{ precisely 2-linked} \\ \qquad\qquad \vdots \\ \leftarrow \text{ precisely } m\text{-linked} \\ \qquad\qquad \vdots \end{array} \end{array}$$

(c) The family of matrices $\{a^\beta_{\zeta,m}\colon \zeta < \kappa,\ 0 < m < \omega,\ \beta < \lambda\}$ is called a κ by λ *independent linked family* w.r.t. $\mathcal{F}$ iff: for each $\beta < \lambda$, the matrix $\{a^\beta_{\zeta,m}\colon \zeta < \kappa,\ 0 < m < \omega\}$ is a linked system w.r.t. $\mathcal{F}$, and:
$\forall b \in \mathcal{F}\ \forall S \in [\lambda]^{<\omega}$ and given, for each $\beta \in S$, some $0 < m_\beta < \omega$ and some $T_\beta \subset \kappa$ of size m_β, we have:

$$b \wedge \bigwedge_{\beta \in S} \bigwedge_{\zeta \in T_\beta} a^\beta_{\zeta,m_\beta} >^* 0_{\mathcal{B}}.$$

Let's say that an element c of $\mathcal{B}$ is *consistent with* $\mathcal{F}$ iff $\forall b \in \mathcal{F}$ $(b \wedge c >^* 0_{\mathcal{B}})$. Note that if $\mathcal{F}$ is nice over $\{b_n\colon n \in \omega\}$, then c is consistent with $\mathcal{F}$ iff $c \notin \mathcal{F}^*$. Given a family $\{a^\beta_{\zeta,m}\colon \zeta < \kappa,\ 0 < m < \omega,\ \beta < \lambda\}$ which is κ by λ independent linked w.r.t. $\mathcal{F}$, (b) above says (among other things) that if we fix one matrix and fix $m > 0$, then the meet of m things from row m of that matrix is consistent with $\mathcal{F}$. For the moment, refer to the meet of m things from row m of a matrix (for any finite $m > 0$) as a "small meet". Then (c) says that if we fix finitely many of the matrices and choose a small meet from each one (letting m depend on the matrix), then the resulting "large meet" is also consistent with $\mathcal{F}$.

Now, in the case that $\mathcal{B} = \mathcal{P}(\omega)$ and each $b_n = \{n\}$, then the definition of *independent linked family* is the same as Kunen's in [7]. Note that here, "$A >^* 0$" just means that A is an infinite subset of ω. From this same paper, we have:

Lemma 3.2. *If $\mathcal{B} = \mathcal{P}(\omega)$, $\forall n$ $(b_n = \{n\})$, and $\mathcal{F}$ is the cofinite filter over ω, then there is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$.*

From this, we get:

Corollary 3.3. *If $\mathcal{B}$ is any complete Boolean algebra, $b_0, b_1, b_2, \ldots$ are disjoint non-0 elements of $\mathcal{B}$, and $\mathcal{F}$ is nice over $\{b_n\colon n \in \omega\}$, then $\mathcal{B}$ contains a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$.*

Proof. Begin with $\{A^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\}$ a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family in $\mathcal{P}(\omega)$ as given in the previous lemma. In $\mathcal{B}$, define $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\}$ in the obvious way: each $a^\beta_{\zeta,m} := \bigvee\{b_n\colon n \in A^\beta_{\zeta,m}\}$. Then clearly any “large meet” X from this collection of matrices will cover infinitely many of the b_n’s. Now consider the nice filter $\mathcal{F}$. By the definition of nice, any $b \in \mathcal{F}$ has non-0 meet with all but finitely many of the b_n’s. So, $X \wedge b$ will have non-0 meet with infinitely many of the b_n’s, i.e., $X \wedge b >^* 0_{\mathcal{B}}$. Therefore X is consistent with $\mathcal{F}$.

Note that for the original matrices over ω, we had by definition that $A^\beta_{\zeta,1} \leqslant A^\beta_{\zeta,2} \leqslant \cdots$ in the strict sense of $\leqslant$, i.e., in $\mathcal{P}(\omega)$. Therefore $a^\beta_{\zeta,1} \leqslant a^\beta_{\zeta,2} \leqslant \cdots$, so the columns of each matrix are increasing. So yes, $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\}$ is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$. □

In the proof of Theorem 1.6, we actually want to begin with a family

$$\left\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\right\} \cup \{x_\gamma\colon \gamma < \mathfrak{c}\}$$

such that the $a^\beta_{\zeta,m}$’s form an independent linked family w.r.t. $\mathcal{F}$ and, for all large meets Y from this linked family,

$$\forall m\ \forall \gamma_1 < \gamma_2 < \cdots < \gamma_m\ \left(Y \wedge x^\pm_{\gamma_1} \wedge x^\pm_{\gamma_2} \wedge \cdots \wedge x^\pm_{\gamma_m} \text{ is consistent with } \mathcal{F}\right).$$

We can get this family as Kunen did in [7]; begin with a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$. Enumerate half of the $\mathfrak{c}$ matrices as $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\}$ and the other half as $\{b^\gamma_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \gamma < \mathfrak{c}\}$. For $\gamma < \mathfrak{c}$, set $x_\gamma = b^\gamma_{0,1}$, i.e., $x_\gamma =$ the first thing from the first row of the γth matrix (anything from the first row will do). Then the collection of $a^\beta_{\zeta,m}$’s and x_γ’s will give us what we want.

Now build up the homomorphism $h = \bigcup_{\alpha<\mathfrak{c}} h_\alpha : \mathcal{B} \twoheadrightarrow \mathcal{A}$ by induction on $\alpha < \mathfrak{c}$. As in [7,9], make h well-defined on the even steps and take care of property $(*)$ on the odd steps. Since $|\mathcal{B}| = \mathfrak{c}$, let $\mathcal{B} = \{b_\alpha\colon \alpha < \mathfrak{c}$ and α is even$\}$. List the decreasing ω-sequences in $\mathcal{B}$ as $\{\vec{d}^\alpha\colon \alpha < \mathfrak{c}$ and α is odd$\}$, with each sequence appearing cofinally often (the first time we consider a particular sequence, it may not yet be relevant). Each $\vec{d}^\alpha$ is of the form $\langle d_0, d_1, \ldots\rangle$, with $d_0 > d_1 > \cdots$.

Using the x_γ’s, we will make h onto right away. Let
- f be any function $f : \{x_\gamma\colon \gamma < \mathfrak{c}\} \twoheadrightarrow \mathcal{A}$.
- g be the constant function $g : \mathcal{F} \to 1_{\mathcal{A}}$.
- $\mathcal{B}_0$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{F} \cup \{x_\gamma\colon \gamma < \mathfrak{c}\}$.
- $h_0 : \mathcal{B}_0 \to \mathcal{A}$ the homomorphic extension of $f \cup g$.
- $\mathcal{F}_0$ the filter $\{b \in \mathcal{B}_0\colon h_0(b) = 1_{\mathcal{A}}\}$ over $\mathcal{B}_0$. Note that $\mathcal{F} \subset \mathcal{F}_0$.
- $I_0 = \emptyset$.

It is easily seen that the family of matrices $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta \in \mathfrak{c}\}$ is independent linked with respect to $\mathcal{F}_0$.

We will define $\mathcal{B}_\alpha, h_\alpha, \mathcal{F}_\alpha, I_\alpha$ $(\alpha < \mathfrak{c})$ such that:

1. For each α, $\mathcal{B}_\alpha$ is a subalgebra of $\mathcal{B}$, $h_\alpha : \mathcal{B}_\alpha \twoheadrightarrow \mathcal{A}$ is a homomorphism, $\mathcal{F}_\alpha = \{b \in \mathcal{B}_\alpha\colon h_\alpha(b) = 1_{\mathcal{A}}\}$, and $I_\alpha \subset \mathfrak{c}$. Each of these increases as α increases.
2. For limit δ, $\mathcal{B}_\delta = \bigcup_{\alpha<\delta} \mathcal{B}_\alpha$, $h_\delta = \bigcup_{\alpha<\delta} h_\alpha$, $\mathcal{F}_\delta = \bigcup_{\alpha<\delta} \mathcal{F}_\alpha$, and $I_\delta = \bigcup_{\alpha<\delta} I_\alpha$.
3. $\forall \alpha < \mathfrak{c}$ $(|I_{\alpha+1} \backslash I_\alpha| < \kappa)$, where κ is the Suslin number of $\mathcal{A}$. This will be explained more later, but the point is that, during our induction, I_α keeps track of the indices of the matrices which we have "used up" before step α. Requirement 3 implies that $\forall \alpha < \mathfrak{c}$ $(|I_\alpha| < \mathfrak{c})$, which in turn ensures that we've never used up the entire family of matrices.
4. For each α, the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta \in \mathfrak{c} \backslash I_\alpha\}$ of unused matrices is an independent linked family w.r.t. $\mathcal{F}_\alpha$.
5. In fact, $\forall c \in \mathcal{B}_\alpha$ $(h_\alpha(c) > 0_{\mathcal{A}} \Rightarrow$ for each large meet Y from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c}$, $0 < m < \omega$, $\beta \in \mathfrak{c} \backslash I_\alpha\}$, $Y \wedge c >^* 0_{\mathcal{B}})$. This condition implies the previous one, since $\mathcal{F}_\alpha = \{b \in \mathcal{B}_\alpha\colon h_\alpha(b) = 1_{\mathcal{A}}\}$.
6. For all $b \in \mathcal{B}$ there is an α s.t. $b \in \mathcal{B}_\alpha$. (Even steps.)
7. For $h := \bigcup_\alpha h_\alpha$, property $(*)$ holds. (Odd steps.)

Most of these conditions obviously hold for $\alpha = 0$, but we should check Condition 5. Since $I_0 = \emptyset$, we need: $\forall c \in \mathcal{B}_0$ $(h_0(c) > 0_{\mathcal{A}} \Rightarrow$ for each large meet Y from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta \in \mathfrak{c}\}$, $Y \wedge c >^* 0_{\mathcal{B}})$. Given $c \in \mathcal{B}_0$, we can write $c = c_1 \vee c_2 \vee \cdots \vee c_n$ where each c_i is a finite meet of things from $\mathcal{F} \cup \mathcal{F}^* \cup \{x_\gamma\colon \gamma < \mathfrak{c}\} \cup \{\widetilde{x}_\gamma\colon \gamma < \mathfrak{c}\}$. Assuming that $h_0(c) > 0_{\mathcal{A}}$, we have $h_0(c_i) > 0_{\mathcal{A}}$ for some i. Then c_i cannot contain anything from $\mathcal{F}^*$, so we can write $c_i = b \wedge x^\pm_{\gamma_0} \wedge x^\pm_{\gamma_1} \wedge \cdots \wedge x^\pm_{\gamma_m}$ where $b \in \mathcal{F}$. By our original choice of the $a^\beta_{\zeta,m}$'s and x_γ's, we have that $\forall d \in \mathcal{F}$ $\forall$ large Y $(Y \wedge d \wedge x^\pm_{\gamma_0} \wedge \cdots \wedge x^\pm_{\gamma_m} >^* 0_{\mathcal{B}})$, so putting b in for d, we get: $\forall$ large Y $(Y \wedge c_i >^* 0_{\mathcal{B}})$. Therefore $\forall$ large Y $(Y \wedge c >^* 0_{\mathcal{B}})$.

We should say a word about condition 3 before moving on. We have specified that κ is the Suslin number of $\mathcal{A}$, i.e., $\kappa =$ the minimal cardinal such that $\mathcal{A}$ has no antichain of size κ. So by a theorem of Tarski, κ is necessarily regular. Furthermore, $\kappa \leqslant \mathfrak{c}$; otherwise, $\mathcal{A}$ has an antichain of size $\mathfrak{c}$ so $\mathcal{A}$ itself is of size at least $2^{\mathfrak{c}}$ (since $\mathcal{A}$ is complete). But we're assuming all along that $\mathcal{A}$ has size $\mathfrak{c}$.

Now we show that the requirement $\forall \alpha < \mathfrak{c}$ $(|I_{\alpha+1} \backslash I_\alpha| < \kappa)$ implies that $\forall \alpha < \mathfrak{c}$ $(|I_\alpha| < \mathfrak{c})$. The proof is by induction on α. Since $I_0 = \emptyset$, the base case is taken care of. The successor steps are easy, so the proof comes down to the limit steps δ. There are two cases; note that in case (ii), we're actually showing that $\forall \alpha < \mathfrak{c}$ $(|I_\alpha| \leqslant \max(|\alpha|, \kappa))$.

Case (i): $\kappa = \mathfrak{c}$. Since κ is regular, we have that $\mathfrak{c}$ is regular. Since I_δ is the union of less than $\mathfrak{c}$ sets, each of size $< \mathfrak{c}$, we know that $|I_\delta| < \mathfrak{c}$.

Case (ii): $\kappa < \mathfrak{c}$. Then $I_\delta = \bigcup_{\alpha<\delta} I_\alpha$ is the union of δ sets, each of size $\leqslant \max(|\delta|, \kappa)$. So $|I_\delta| \leqslant \max(|\delta|, \kappa) < \mathfrak{c}$.

Now to finish the proof of the theorem, it remains to show that the odd steps, even steps, and limit steps can be done while preserving (or in the case of requirements 6 and 7, accomplishing) each of the conditions listed above. Condition 2 tells us how to handle the limit steps, and checking that the other relevant requirements still hold after a limit step is easy. So, we now show how to handle the successor steps.

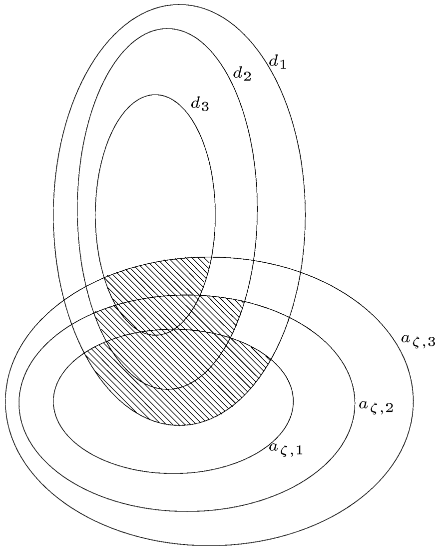
Odd steps

Notation. If $X \subset \mathcal{B}$, then $((X))$ is the subalgebra of $\mathcal{B}$ generated by X. If $\mathcal{C}$ is a subalgebra of $\mathcal{B}$, $g:\mathcal{C} \to \mathcal{A}$ is a homomorphism, and $b \in \mathcal{B}$, then $h^+(b) = \bigwedge\{h(c)\colon\ c \in \mathcal{C} \text{ and } c \geqslant b\}$ and $h^-(b) = \bigvee\{h(c)\colon\ c \in \mathcal{C} \text{ and } c \leqslant b\}$.

Now, we have $h_\alpha : \mathcal{B}_\alpha \twoheadrightarrow \mathcal{A}$ and $\mathcal{F}_\alpha$ such that all of the appropriate inductive hypotheses hold. We are considering $\vec{d}^\alpha = \langle d_0, d_1, \ldots\rangle$, with $d_0 > d_1 > \cdots$. Assume that $\forall i\ (d_i \in \mathcal{F}_\alpha)$ (otherwise, let $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha$ and $h_{\alpha+1} = h_\alpha$).

Fix a matrix $\{a^\beta_{\zeta,m}\colon\ \zeta < \mathfrak{c},\ 0 < m < \omega\}$ such that $\beta \in \mathfrak{c}\setminus I_\alpha$. This is the matrix that we will "use up" during this step. From now on we'll leave off the superscript β. Define y_ζ for $\zeta < \mathfrak{c}$ by:

$$y_\zeta = \bigvee_{1 \leqslant j < \omega} \big(a_{\zeta,j} \wedge d_j\big).$$



We claim that these y_ζ's satisfy the second condition required by Property $(*)$, i.e., $\forall m\, \forall \zeta_1 < \zeta_2 < \cdots < \zeta_m < \mathfrak{c}\ (y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m} \leqslant^* d_m)$. To check this, fix m and fix $\zeta_1 < \cdots < \zeta_m < \mathfrak{c}$. For each i, we have

$$y_{\zeta_i} = \Big(\bigvee_{1 \leqslant j < m} \big(a_{\zeta_i,j} \wedge d_j\big)\Big) \vee \Big(\bigvee_{j \geqslant m} \big(a_{\zeta_i,j} \wedge d_j\big)\Big).$$

In the first join, since the $a_{\zeta_i,j}$'s increase as j increases, each $a_{\zeta_i,j} \wedge d_j$ is $\leqslant a_{\zeta_i,m-1}$. And in the second join, since the d_j's decrease as j increases, each $a_{\zeta_i,j} \wedge d_j$ is $\leqslant d_m$. Therefore $\forall i\ (y_{\zeta_i} \leqslant a_{\zeta_i,m-1} \vee d_m)$. Taking the meet of the y_{ζ_i}'s for $1 \leqslant i \leqslant m$, we have:

$$\big(y_{\zeta_1} \wedge y_{\zeta_2} \wedge \cdots \wedge y_{\zeta_m}\big) \leqslant d_m \vee \big(a_{\zeta_1,m-1} \wedge a_{\zeta_2,m-1} \wedge \cdots \wedge a_{\zeta_m,m-1}\big).$$

Since $a_{\zeta_1,m-1} \wedge a_{\zeta_2,m-1} \wedge \cdots \wedge a_{\zeta_m,m-1}$ is an intersection of m things from row $m-1$, it has non-0 meet with just finitely many b_n's. So $(y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m}) \backslash d_m$ has non-0 meet with just finitely many b_n's. So $y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m} \leqslant^* d_m$.

Now we wish to set $\mathcal{B}_{\alpha+1} = ((\mathcal{B}_\alpha \cup \{y_\zeta\colon \zeta < \mathfrak{c}\}))$ and extend h_α to $h_{\alpha+1}$ by specifying that $\forall \zeta\ h_{\alpha+1}(y_\zeta) = 1_{\mathcal{A}}$, so that $\mathcal{F}_{\alpha+1} = \langle \mathcal{F}_\alpha \cup \{y_\zeta\colon \zeta < \mathfrak{c}\}\rangle$. Also, we will have $I_{\alpha+1} = I_\alpha \cup \{\beta\}$. But we need to check that the inductive hypotheses will still hold. There are only two which might cause trouble.

First, inductive hypothesis 1: is it consistent to extend h_α *homomorphically* by sending each y_ζ to $1_{\mathcal{A}}$? By Sikorski's Extension Theorem (see [8, Theorem 33.1]), since $\mathcal{A}$ is complete, for a fixed ζ we can extend h_α by sending y_ζ to anything between $h_\alpha^-(y_\zeta)$ and $h_\alpha^+(y_\zeta)$. So, to send y_ζ to $1_{\mathcal{A}}$, we need that $h_\alpha^+(y_\zeta) = 1_{\mathcal{A}}$. But since we want to send all the y_ζ's to $1_{\mathcal{A}}$ at the same time, we actually need that $\forall m\, \forall \zeta_1 < \zeta_2 < \cdots < \zeta_m$ $(h_\alpha^+(y_{\zeta_1} \wedge y_{\zeta_2} \wedge \cdots \wedge y_{\zeta_m}) = 1_{\mathcal{A}})$.

Write $y := y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m}$. By definition, $h_\alpha^+(y) = \wedge\{h_\alpha(c)\colon c \in \mathcal{B}_\alpha \text{ and } y \leqslant c\}$. Assume that this is less than $1_{\mathcal{A}}$ (for a contradiction). Then $\exists c \in \mathcal{B}_\alpha$ such that $c \geqslant y$ and $h_\alpha(c) < 1_{\mathcal{A}}$. So, $y \wedge \widetilde{c} = 0_{\mathcal{B}}$ and $h_\alpha(\widetilde{c}) > 0_{\mathcal{A}}$. Since $h_\alpha(d_m) = 1_{\mathcal{A}}$ (remember that each d_m is in $\mathcal{F}_\alpha$), we have $h_\alpha(\widetilde{c} \wedge d_m) = h_\alpha(\widetilde{c}) \wedge h_\alpha(d_m) = h_\alpha(\widetilde{c}) \wedge 1_{\mathcal{A}} > 0_{\mathcal{A}}$. So by inductive hypothesis 5 at step α,

$$a_{\zeta_1,m} \wedge a_{\zeta_2,m} \wedge \cdots \wedge a_{\zeta_m,m} \wedge \widetilde{c} \wedge d_m >^* 0_{\mathcal{B}}.$$

But note that $\forall i\ (y_{\zeta_i} \geqslant a_{\zeta_i,m} \wedge d_m)$, so

$$y = (y_{\zeta_1} \wedge y_{\zeta_2} \wedge \cdots \wedge y_{\zeta_m}) \geqslant (a_{\zeta_1,m} \wedge a_{\zeta_2,m} \wedge \cdots \wedge a_{\zeta_m,m} \wedge d_m).$$

Therefore $y \wedge \widetilde{c} >^* 0_{\mathcal{B}}$. But this contradicts the assumption that $y \wedge \widetilde{c} = 0_{\mathcal{B}}$.

Second, inductive hypothesis 5: check that $\forall c \in \mathcal{B}_{\alpha+1}$ $(h_{\alpha+1}(c) > 0_{\mathcal{A}} \Rightarrow \forall$ large Y from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c}, 0 < m < \omega, \beta \in \mathfrak{c}\backslash I_{\alpha+1}\}$, $Y \wedge c >^* 0_{\mathcal{B}})$. Write $c = c_1 \vee c_2 \vee \cdots \vee c_n$ where each c_i is a finite meet of things from $\mathcal{B}_\alpha \cup \{y_\zeta\colon \zeta < \mathfrak{c}\} \cup \{\widetilde{y_\zeta}\colon \zeta < \mathfrak{c}\}$. Assuming that $h_{\alpha+1}(c) > 0_{\mathcal{A}}$, we have $h_{\alpha+1}(c_i) > 0_{\mathcal{A}}$ for some i. This c_i cannot contain a $\widetilde{y_\zeta}$, so write $c_i = b \wedge y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m}$ where $b \in \mathcal{B}_\alpha$. Then $h_\alpha(b) > 0_{\mathcal{A}}$, so $h_\alpha(b \wedge d_m) > 0_{\mathcal{A}}$.

Now let Y be a large meet from our family of matrices as given above. Want: $Y \wedge c_i >^* 0_{\mathcal{B}}$. But since each $y_{\zeta_i} \geqslant a_{\zeta_i,m} \wedge d_m$, we have:

$$Y \wedge c_i = Y \wedge b \wedge y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m} \geqslant Y \wedge b \wedge a_{\zeta_1,m} \wedge \cdots \wedge a_{\zeta_m,m} \wedge d_m = X \wedge b \wedge d_m,$$

where X is a large meet from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta \in \mathfrak{c}\backslash I_\alpha\}$. Since $h_\alpha(b \wedge d_m) > 0_{\mathcal{A}}$, the inductive hypothesis at step α implies $X \wedge b \wedge d_m >^* 0_{\mathcal{B}}$, so we are done.

Even steps

We have $h_\alpha : \mathcal{B}_\alpha \twoheadrightarrow \mathcal{A}$ and $\mathcal{F}_\alpha$ such that all of the appropriate inductive hypotheses hold. We have $x \in \mathcal{B}$ and we wish to extend h_α to $h_{\alpha+1}$ so that $h_{\alpha+1}(x)$ is defined. Without lost of generality $x \notin \mathcal{B}_\alpha$ (otherwise, let $\mathcal{B}_{\alpha+1} = \mathcal{B}_\alpha$ and $h_{\alpha+1} = h_\alpha$).

Again by Sikorski's Extension Theorem, we can extend h_α to a homomorphism $h_{\alpha+1} : \mathcal{B}_{\alpha+1} \to \mathcal{A}$ by setting $h_{\alpha+1}(x) =$ anything between $h_\alpha^-(x)$ and $h_\alpha^+(x)$. Here, $\mathcal{B}_{\alpha+1}$

would be $((\mathcal{B}_\alpha \cup \{x\}))$ and $I_{\alpha+1}$ would just be I_α. However, to preserve the inductive hypothesis 5, we need to be more careful. Need: $\forall c \in \mathcal{B}_{\alpha+1}$ $(h_{\alpha+1}(c) > 0_{\mathcal{A}} \Rightarrow \forall$ large meets Y from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c}, 0 < m < \omega, \beta \in \mathfrak{c}\setminus I_{\alpha+1}\}, Y \wedge c >^* 0_{\mathcal{B}})$.

In general, meeting this requirement will take more work than simply setting $h_{\alpha+1}(x)$ arbitrarily between $h^-_\alpha(x)$ and $h^+_\alpha(x)$. Instead, we will build $h_{\alpha+1}$ through a series of intermediate induction steps. To begin, set $h^0 = h_\alpha$, $\mathcal{B}^0 = \mathcal{B}_\alpha$, and $I^0 = I_\alpha$. We will construct increasing $h^\gamma, \mathcal{B}^\gamma$, and I^γ such that for each γ, h^γ is a homomorphism from $\mathcal{B}^\gamma$ to $\mathcal{A}$ which, with I^γ, satisfies inductive hypothesis 5.

During the intermediate step $\gamma + 1$, we try to complete $h_{\alpha+1}$ by setting $h_{\alpha+1}(x) = (h^\gamma)^+(x)$, the current maximal possible element of $\mathcal{A}$. If it is not possible to do this without violating inductive hypothesis 5, then there is some "problem" element in $((\mathcal{B}^\gamma \cup \{x\}))$ which witnesses the violation. We take care of this problem element by setting $h^{\gamma+1}(Y) = 1_{\mathcal{A}}$ for some appropriate large meet Y, then updating $I^{\gamma+1}$ to throw out the matrices associated with Y. Eventually, we will have taken care of all of the problem elements. The details are as below.

- *Successor steps*, $\gamma + 1$: If it is okay to extend h^γ to $h_{\alpha+1}$ on $\mathcal{B}_{\alpha+1} = ((\mathcal{B}^\gamma \cup \{x\}))$ by setting $h_{\alpha+1}(x) = (h^\gamma)^+(x)$ and $I_{\alpha+1} = I^\gamma$, then do so and stop.

Otherwise $\exists c \in ((\mathcal{B}^\gamma \cup \{x\}))$ which presents a problem; i.e., sending x to $(h^\gamma)^+(x)$ would result in sending c to something $> 0_{\mathcal{A}}$, but there is a large Y from the family $\{a^\beta_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta \in \mathfrak{c}\setminus I^\gamma\}$ such that $c \wedge Y =^* 0_{\mathcal{B}}$.

Now define $\mathcal{B}^{\gamma+1} = ((\mathcal{B}^\gamma \cup \{Y\}))$ and extend h^γ to $h^{\gamma+1}$ by setting $h^{\gamma+1}(Y) = 1_{\mathcal{A}}$. Also, set $I^{\gamma+1} = I^\gamma$ plus the indices of the matrices associated with Y. It is easy to see that $h^{\gamma+1}, \mathcal{B}^{\gamma+1}$, and $I^{\gamma+1}$ satisfy Condition 5. Furthermore, when we finally define the homomorphism $h_{\alpha+1}$ to include x, the element c will no longer be a problem. Why? Since $c \wedge Y =^* 0_{\mathcal{B}}$, we have $c \leqslant^* \widetilde{Y}$ so $c < \widetilde{Y} \vee$ (some finite join of the b_n's) $\vee (\bigvee_n b_n)^{\sim}$. Therefore $(h^{\gamma+1})^+(c) \leqslant h^{\gamma+1}(\widetilde{Y}) \vee 0_{\mathcal{A}} \vee 0_{\mathcal{A}} = 0_{\mathcal{A}}$. So $h_{\alpha+1}(c)$ will have to be $0_{\mathcal{A}}$, and condition 5 for c will be vacuously true.

Now technically we are done with step $\gamma + 1$, but for later reasons, before moving on we will prove the following.

Claim. $(h^{\gamma+1})^+(x) < (h^\gamma)^+(x)$.

Proof. Clearly $(h^{\gamma+1})^+(x) \leqslant (h^\gamma)^+(x)$, since $h^{\gamma+1}$ is an extension of h^γ. So, we just need to show that equality is not possible. To do this, we need to go into more detail concerning the problem element $c \in ((\mathcal{B}^\gamma \cup \{x\}))$ which we took care of above. Write $c = (d \wedge x) \vee (e \wedge \widetilde{x})$ where $d, e \in \mathcal{B}^\gamma$.

Let Y be the large meet from above. Since $c \wedge Y =^* 0_{\mathcal{B}}$, we have $d \wedge x \wedge Y =^* 0_{\mathcal{B}}$. So, $x \leqslant^* \tilde{d} \vee \widetilde{Y}$. Therefore $x < \tilde{d} \vee \widetilde{Y} \vee$ (some finite join of the b_n's) $\vee (\bigvee_n b_n)^{\sim}$. So $(h^{\gamma+1})^+(x) \leqslant h^{\gamma+1}(\tilde{d}) \vee 0_{\mathcal{A}} \vee 0_{\mathcal{A}} \vee 0_{\mathcal{A}} = h^\gamma(\tilde{d})$.

Now to finish the claim, we show that $(h^\gamma)^+(x) \nleqslant h^\gamma(\tilde{d})$. Since $c \wedge Y =^* 0_{\mathcal{B}}$, we have $e \wedge \widetilde{x} \wedge Y =^* 0_{\mathcal{B}}$. So $\forall f \in \mathcal{B}^\gamma (f \leqslant \widetilde{x} \Rightarrow (e \wedge f \wedge Y =^* 0_{\mathcal{B}}))$. By Condition 5 at step γ, we have that $\forall f \in \mathcal{B}^\gamma (f \leqslant \widetilde{x} \Rightarrow (h^\gamma(e) \wedge h^\gamma(f) = 0_{\mathcal{A}}))$. Therefore $h^\gamma(e) \wedge (h^\gamma)^-(\widetilde{x}) = 0_{\mathcal{A}}$.

Now, remember that sending x to $(h^\gamma)^+(x)$ results in sending c to something bigger than $0_{\mathcal{A}}$, and note that it also results in sending $\widetilde{x}$ to $(h^\gamma)^-(\widetilde{x})$. Since $c = (d \wedge x) \vee (e \wedge \widetilde{x})$, the conclusion of the last paragraph tells us that $h^\gamma(d) \wedge (h^\gamma)^+(x) > 0_{\mathcal{A}}$. So, as desired, $(h^\gamma)^+(x) \nleqslant h^\gamma(\tilde{d})$. This completes the claim. $\square$

• *Limits*, δ: Get $\mathcal{B}^\delta$, h^δ, and I^δ using unions.

Since the $(h^\gamma)^+(x)$'s are strictly decreasing, this process will have to end at some point. So we *will* get our $\mathcal{B}_{\alpha+1}$, $h_{\alpha+1}$, and $I_{\alpha+1}$. And by our construction, most of the appropriate inductive hypotheses will clearly hold for $\alpha + 1$. However, we do need to check that we're preserving inductive hypothesis 3. (This is not so obvious as it was for the odd steps.)

We need to show that $|I_{\alpha+1} \backslash I_\alpha| < \kappa$. Remember that κ is the Suslin number of $\mathcal{A}$. In finding $I_{\alpha+1}$, we have formed the chain $(h^0)^+(x) > (h^1)^+(x) > \cdots > (h^\gamma)^+(x) > \cdots$. This chain creates an antichain of the same size, and therefore must have size less than κ. At each intermediate step (represented by an item in the chain), only finitely many matrices are used, so the overall number of new matrices thrown out during an even step is $\leqslant \omega *$ (the size of the chain), which is $< \kappa$ as desired.

4. A small generalization

In Theorem 1.6, $\mathcal{A}$ and $\mathcal{B}$ are both assumed to be complete. The completeness of $\mathcal{A}$ is needed whenever Sikorski's Extension Theorem is used to extend homomorphisms from a subalgebra of $\mathcal{B}$ into $\mathcal{A}$. However, all we really need for $\mathcal{B}$ is that it have the countable separation property. Before stating the more generalized theorem, we need to generalize the definition of *nice filter*.

Definition 4.1. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of the boolean algebra $\mathcal{B}$. Then a filter $\mathcal{F}$ over $\mathcal{B}$ is *nice* over $\{b_n\colon n \in \omega\}$ iff:

(1) $\forall b \in \mathcal{B}\ ((\forall n\ b \wedge b_n = 0_{\mathcal{B}}) \Rightarrow \tilde{b} \in \mathcal{F})$.
(2) $\forall n < \omega\ (\widetilde{b_n} \in \mathcal{F})$.
(3) $\forall b \in \mathcal{F}\ (\{n\colon b \wedge b_n = 0_{\mathcal{B}}\}$ is finite$)$.

The only difference between this definition and the previous one is in the first item, where we can no longer use the infinite join $\bigvee_n b_n$. In the definition of *misses countable sets*, as in the statement of the theorem, we also need to replace the $\bigvee_n b_n$. In both of these cases, it suffices to set $\mathcal{Z} = \mathrm{cl}(\bigcup_n N_{b_n}) \setminus \bigcup_n N_{b_n}$ rather than $\mathcal{Z} = N_{\bigvee_n b_n} \setminus \bigcup_n N_{b_n}$. (When $\mathcal{B}$ is complete, these are exactly the same sets.) The new theorem is as follows.

Theorem 4.2. *Assume that $\mathcal{A}$ is complete, $\mathcal{B}$ has the countable separation property (c.s.p.), and both have cardinality $\mathfrak{c}$. Let $b_0, b_1, b_2, \ldots$ be disjoint non-0 elements of $\mathcal{B}$ and set $\mathcal{Z} = \mathrm{cl}(\bigcup_n N_{b_n}) \setminus \bigcup_n N_{b_n}$. Let $\mathcal{F}$ be a nice filter over $\{b_n\colon n \in \omega\}$. Then $\mathrm{st}(\mathcal{A})$ is homeomorphic to some $\mathfrak{c}$-OK subset $\mathcal{K}$ of $\mathcal{Z}$, and in addition $\mathcal{K} \subset \mathcal{K}_{\mathcal{F}}$.*

Before proving the revised theorem, note that the definition of *independent linked family w.r.t.* $\mathcal{F}$ needs no modification; however, in Corollary 3.3, when we found a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. the nice filter $\mathcal{F}$, we did assume that $\mathcal{B}$ was complete. So, we need to re-prove this corollary for the case when $\mathcal{B}$ just has the c.s.p.

Claim 4.3. *If $\mathcal{B}$ has the c.s.p., $b_0, b_1, b_2, \ldots$ are disjoint non-*0 *elements of $\mathcal{B}$, and $\mathcal{F}$ is nice over $\{b_n\colon n \in \omega\}$, then $\mathcal{B}$ contains a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$.*

Proof. Using Lemma 3.2, begin with $\{A^{\beta}_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega,\ \beta < \mathfrak{c}\}$ over $\mathcal{P}(\omega)$, an independent linked family w.r.t. the cofinite filter. We wish to define $\{a^{\beta}_{\zeta,m}\colon \zeta < \mathfrak{c}, 0 < m < \omega,\ \beta < \mathfrak{c}\}$ over $\mathcal{B}$ such that this family is independent linked w.r.t. $\mathcal{F}$. This time we can't simply define these elements using infinite joins. Instead, using the c.s.p., set $a^{\beta}_{\zeta,m} =$ some element of $\mathcal{B}$ which satisfies:

$$\forall n\big[\big(n \in A^{\beta}_{\zeta,m} \Rightarrow a^{\beta}_{\zeta,m} \geqslant b_n\big) \text{ and } \big(n \notin A^{\beta}_{\zeta,m} \Rightarrow a^{\beta}_{\zeta,m} \wedge b_n = 0_{\mathcal{B}}\big)\big].$$

Note that, generally, these elements are not uniquely determined. This is not a problem, and as in the proof of Corollary 3.3 it is easily seen that the matrices $\{a^{\beta}_{\zeta,m}\colon \zeta < \mathfrak{c},\ m < \omega,\ \beta < \mathfrak{c}\}$ satisfy most of what is required to be an independent linked family w.r.t. the filter $\mathcal{F}$. However, since we do not know what the $a^{\beta}_{\zeta,m}$'s look like outside the b_n's, it is not certain (or even likely) that for each ζ, $a^{\beta}_{\zeta,1} \leqslant a^{\beta}_{\zeta,2} \leqslant a^{\beta}_{\zeta,3} \leqslant \cdots$, as is required.

To fix this, we give a slight redefinition of the $a^{\beta}_{\zeta,m}$'s. For each ζ and m, replace $a^{\beta}_{\zeta,m}$ with the finite join $a^{\beta}_{\zeta,1} \vee a^{\beta}_{\zeta,2} \vee \cdots \vee a^{\beta}_{\zeta,m}$. It is clear that in the new matrices, each column is increasing. And since $A^{\beta}_{\zeta,1} \leqslant A^{\beta}_{\zeta,2} \leqslant \cdots$, the new $a^{\beta}_{\zeta,m}$'s are different from the old ones only outside the b_n's, so they still satisfy $\forall n[(n \in A^{\beta}_{\zeta,m} \Rightarrow a^{\beta}_{\zeta,m} \geqslant b_n)$ and $(n \notin A^{\beta}_{\zeta,m} \Rightarrow a^{\beta}_{\zeta,m} \wedge b_n = 0_{\mathcal{B}})]$. Therefore the other requirements for independent linked family still hold. $\square$

Proof of Theorem 4.2. The proof of the generalized theorem follows almost exactly as before. The differences occur when finding/defining various elements inside of $\mathcal{B}$, since here we cannot use infinite joins to do so. We have already taken care of the first modification, which concerns finding the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t. $\mathcal{F}$.

The only other modification occurs during the odd steps of the induction, when we need to define the y_ζ's and check that they behave as they should. We have a sequence $d_0 > d_1 > d_2 > \cdots$ from $\mathcal{B}_\alpha$ such that $\forall i\ (d_i \in \mathcal{F}_\alpha)$, and a fixed matrix $\{a_{\zeta,m}\colon \zeta < \mathfrak{c},\ 0 < m < \omega\}$ which we have not used yet.

Now define y_ζ $(\zeta < \mathfrak{c})$ as follows: using the c.s.p., choose y_ζ to be an element of $\mathcal{B}$ which satisfies:

$$\forall m \geqslant 1\ \big[\big(y_\zeta \geqslant a_{\zeta,m} \wedge d_m\big) \text{ and } \big(y_\zeta \wedge \big(a_{\zeta,m} \vee d_{m+1}\big)^{\sim} = 0_{\mathcal{B}}\big)\big].$$

Note: this is the same as saying that y_ζ lies above each $a_{\zeta,m} \wedge d_m$ and below each $a_{\zeta,\ell} \vee d_{\ell+1}$. We need to check that this definition of y_ζ is legal, i.e., that $\forall m\, \forall \ell\ (a_{\zeta,m} \wedge d_m \leqslant a_{\zeta,\ell} \vee d_{\ell+1})$.

Case $m \leqslant \ell$: yes; $a_{\zeta,m} \wedge d_m \leqslant a_{\zeta,m} \leqslant a_{\zeta,\ell} \leqslant a_{\zeta,\ell} \vee d_{\ell+1}$.

Case $m > l$: yes; $a_{\zeta,m} \wedge d_m \leqslant d_m \leqslant d_{\ell+1} \leqslant a_{\zeta,\ell} \vee d_{\ell+1}$.

Do these y_ζ's get us what we want? Is it true that $\forall m \, \forall \zeta_1 < \zeta_2 < \cdots < \zeta_m < \mathfrak{c}$ $(y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m} \leqslant^* d_m)$? For each i between 1 and m, the definition of y_{ζ_i} yields $y_{\zeta_i} \leqslant a_{\zeta_i,\ell} \vee d_{\ell+1}$ for each ℓ. Putting in $m-1$ for ℓ, we get $y_{\zeta_i} \leqslant a_{\zeta_i,m-1} \vee d_m$. But this is exactly the fact that we used in the previous proof to show that $y_{\zeta_1} \wedge \cdots \wedge y_{\zeta_m} \leqslant^* d_m$, so the same reasoning works here.

Now we wish to set $h_{\alpha+1}(y_\zeta) = 1_{\mathcal{A}}$ for each $\zeta < \mathfrak{c}$. It remains to check that the inductive hypotheses will still hold, in particular hypotheses 1 and 5 as before. In the previous proof, these calculations used that fact that $\forall m \, \forall \zeta$ $(y_\zeta \geqslant a_{\zeta,m} \wedge d_m)$, and this is certainly true here as well. So, no update is needed. $\square$

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